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Linear Differential Equations in Banach Spaces

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LINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract

A differential equation of the form $du(t)/dt = A(t)u(t) + f(t)$ in a Banach space, with non-bounded linear operators $A(t)$, is investigated. Some uniqueness and existence theorems are proved. It is hoped that the results may be applicable to linear partial differential equations of parabolic or hyperbolic type.

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1. Introduction

This paper is concerned with the solution of a Cauchy problem in an abstract linear space. Consider a linear differential equation

$$(1.1) \qquad \frac{du(t)}{dt} = A(t)u(t) + f(t), \qquad a \leq t \leq b,$$

and the associated homogeneous equation

$$(1.1a) \qquad \frac{du(t)}{dt} = A(t)u(t).$$

Here t is a real variable; the unknown function u(t) and the given function f(t) take values in a Banach space X; and A(t) is a given function whose values are linear operators in X. We wish to find the solution of (1.1) or (1.1a) when the initial value u(a) is prescribed. The problem is rather trivial if the operators A(t) are bounded* and, say, strongly continuous in t, for then the solution can be obtained by a standard iteration procedure. But we are mainly interested in the case where the A(t) are unbounded, so that their domains are proper subsets of X. In such a case the existence of the solution is by no means obvious.

The Schroedinger equation in quantum mechanics is expressed exactly in the form (1.1a) with X a Hilbert space and with the operators iA(t) self-adjoint. It is also obvious that linear partial differential equations of parabolic type which are of the first order in the time variable t can be written in the form (1.1). Our study is primarily intended for application to such problems. It is to be noted, however, that a large number of hyperbolic differential equations can also be reduced to the form (1.1) (see e.g., [1]), at least formally, by introducing a suitable space X as the set of 'column-vector functions', in which the corresponding 'matrix-functions' $A(t)$ operate. We hope that our results may be applicable at least to some of these problems.

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* Bounded linear operators are understood to be defined everywhere in X, unless otherwise stated.

In this paper we shall present some uniqueness and existence theorems for the solution of (1.1), and the proofs of these theorems are outlined in Sections 3 and 4*.

The formal differential equation (1.1) admits various interpretations. The derivative $du(t)/dt$ may be taken either in the strong or weak sense, and the equality may be required for all t or, e.g., for almost all t (cf. Lions[4]). Again, it may be regarded as an operational equation $[d/dt - A(t)]u(t) = f(t)$ in some linear space consisting of vector-valued functions $u(t)$, $a \leq t \leq b$, and in this interpretation the operator $d/dt - A(t)$ may again be taken in some extended sense, either strong or weak.**

In the present paper we take (1.1) in the simplest sense, however. We require that $du(t)/dt$ exist as a strong derivative in X; that $A(t)u(t)$ make sense; that (1.1) hold for all t in the open interval $a < t < b$; and that $u(t)$ be strongly continuous in the closed interval. In particular, $u(t)$ must remain in the instantaneous domain $\mathcal{D}[A(t)]$ of $A(t)$ - a rather severe requirement in view of the fact that $\mathcal{D}[A(t)]$ is in general of first category in X. However, this is a customary requirement in mixed initial- and boundary-value problems for differential equations.

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* Complete proofs will be published elsewhere. The reader is also referred to an earlier paper[3] by the author, where the same problem is treated under more restrictive assumptions than those used in the present paper.

** Cf. Friedrichs[1], where reference to other related papers may be found.

2. Semi-group theory

It is instructive first to consider the case in which $A(t) = A$ is independent of t. Then the solution of (1.1) is formally given by

$$u(t) = \exp\left[(t-a)A\right]u(a) + \int_a^t \exp\left[(t-s)A\right]f(s)ds. \tag{2.1}$$

The question of defining the exponential function $\exp(tA)$ is exactly the problem of generating a one-parameter semi-group; this has been solved by Hille [2], Yosida [6], Phillips [5] and others.

In the Hille-Yosida theorem the 'infinitesimal generator' A is assumed to satisfy the following conditions: 1) that A be a closed linear operator in X with a dense domain; and 2) that all positive real numbers belong to the resolvent set of A, and that

$$\| (\lambda I - A)^{-1} \| \leq \lambda^{-1}, \qquad \lambda > 0. \tag{2.2}$$

In what follows, the class of operators A satisfying these conditions will be denoted by (S). The Hille-Yosida theorem states that for each $A \in (S)$ there exists a unique operator-valued function $\exp(tA)$ defined for $t \geq 0$ with these properties:

$$\left\{ \begin{array}{l} \exp(tA) \text{ is bounded and strongly continuous} \\ \text{in } t \text{ with } \exp(0A) = I; \end{array} \right. \tag{2.3}$$

$$\exp\left[(t+s)A\right] = \exp(tA)\exp(sA); \tag{2.4}$$

$$\left\{ \begin{array}{l} \exp(tA) \text{ maps the domain } \mathcal{D}(A) \text{ into itself; and for each } u \in \mathcal{D}(A), \\ \exp(tA)u \text{ is strongly differentiable, with} \\ \qquad (d/dt)\exp(tA)u = A\exp(tA)u = \exp(tA)Au; \end{array} \right. \tag{2.5}$$

$$\exp(tA) \text{ is permutable with the resolvent } (\lambda I - A)^{-1}. \tag{2.6}$$

According to Phillips [5], one may replace the inequality (2.2) by the weaker condition*

$$(2.7) \qquad \|(\lambda I - A)^{-n}\| \leq k\lambda^{-n}, \qquad n = 1,2,\ldots, \qquad \lambda > 0,$$

where k is independent of n and λ. The class of operators A satisfying these conditions will be denoted by (S"). Of course $(S'') \supset (S)$. The function exp(tA) can still be defined for each $A \in (S'')$ so as to retain the properties (2.3) to (2.6). Furthermore, Phillips [5] has proved that (2.1) is a solution of (1.1) if f(t) is strongly continuously differentiable. It may be remarked that (2.1) is also a solution of (1.1) if $f(t) \in \mathcal{D}(A)$ for all t and if Af(t) and f(t) are strongly continuous in t.

It is natural to attempt to use the condition $A \in (S'')$ in the general case where A(t) actually depends on t. But it appears that the class (S") is too wide to work with, so we introduce a class (S') of infinitesimal generators lying between (S) and (S"). A linear operator A belongs by definition to (S') if there exists a bounded linear operator Q with a bounded inverse such that

$$\bar{A} = QAQ^{-1}$$

belongs to (S). This implies that Q maps $\mathcal{D}(A)$ onto $\mathcal{D}(\bar{A})$ in a one-to-one fashion. It is easily seen that $(S) \subset (S') \subset (S'')$. The fact that one can define the function exp(tA) for $A \in (S')$ with the properties (2.3) to (2.6) can be established without relying on Phillips' results. We need only put

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* Actually the condition given by Phillips is more general than (2.7) in that the right-hand side of his equation contains $(\lambda-\omega)^{-n}$ instead of λ^{-n}. The difference is not essential, however, since either form can be reduced to the other by using $A + \omega I$ instead of A. We have chosen the form (2.7) for convenience of comparison with the case (2.2).

$$\exp(tA) = Q^{-1} \exp(t\bar{A})\, Q, \tag{2.8}$$

where $\exp(t\bar{A})$ is defined by Hille-Yosida theorem. Note also that since $(\lambda I-A)^{-1} = Q^{-1}(\lambda I-\bar{A})^{-1}Q$, all positive real numbers belong to the resolvent set of A.

3. Uniqueness theorem

After these preliminaries we return to the general case with time-dependent $A(t)$ and introduce the following assumption.

Assumption 1. For each t, $a \leq t \leq b$, the operator $A(t)$ belongs to class (S'), so that there is a bounded linear operator $Q(t)$ with a bounded inverse $Q(t)^{-1}$ such that $\bar{A}(t) = Q(t)A(t)Q(t)^{-1} \in (S)$. Moreover $Q(t)$ is strongly continuously differentiable, that is, the strong derivative[*] $\dot{Q}(t) = dQ(t)/dt$ exists and is strongly continuous.

Theorem 1: Under Assumption 1, the solution of (1.1) is uniquely determined by the initial value $u(a)$.

This is an immediate consequence of the following lemma.

Lemma 1: Let $u(t)$ be strongly continuous for $a \leq t \leq b$. Further, let the strong right-hand derivative $D^+u(t)$ exist, and let $u(t) \in \mathcal{D}[A(t)]$ and

$$D^+u(t) = A(t)u(t)$$

for $a < t < b$. Then there is a constant c depending only on $A(t)$ such that $||u(t)|| \leq c\,||u(a)||$ for $a \leq t \leq b$.

This lemma can be reduced to a corresponding theorem, proved in the earlier paper [3], for the case where $A(t) \in (S)$, if we introduce a new unknown

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[*] The results would be the same if we assume that $\dot{Q}(t)$ is a weak derivative.

$$y(t) = \exp(-ht)Q(t)\mu(t)$$

with a suitable constant h.

4. Existence theorem

In order to prove the existence of the solution of (1.1) for a given initial value u(a), we need some additional assumptions.* In general, it cannot be expected that a solution exists for all $u(a) \in X$, since this is not the case even if $A(t) = A$ is independent of t (see (2.5)). Thus we seek a theorem which will ensure the existence of a solution for each u(a) belonging to $\mathcal{D}[A(a)]$. Essentially this implies that we require the validity of the differential equation (1.1) not only in the open interval $a < t < b$ but also at the initial time $t = a$.**

In addition to Assumption 1, we make the following assumption:

Assumption 2. The operator A(t) changes smoothly with t in the following sense. There exists a bounded operator-valued function R(t) with a bounded inverse $R(t)^{-1}$ and with a strongly continuous second derivative*** $\ddot{R}(t)$ such that the operator $\bar{\bar{A}}(t) = R(t)A(t)R(t)^{-1}$ has a domain $\mathcal{D}$ independent of t. Then it follows that the operator $B(t) = [I - \bar{\bar{A}}(t)][I - \bar{\bar{A}}(a)]^{-1}$ is bounded for each t. It is further assumed that B(t) is strongly continuously differentiable.

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* The second example of Section 5 shows that the change of A(t) must be sufficiently smooth in some sense.

** This is an essential restriction of our theory. There are many problems in which the initial value may be arbitrary and yet the differential equation is satisfied in the open interval. This is usually the case for classical parabolic equations. However, the condition $u(a) \in D[A(a)]$ is indispensable in the case of the Schroedinger equation.

*** Again, the result would be the same if we assume that $\ddot{R}(t)$ is a weak derivative.

Let us examine this assumption and its immediate consequences. The operator $R(t)$ maps $\mathcal{D}[A(t)]$ onto $\mathcal{D}$ in a one-to-one fashion. Since by Assumption 1 all positive real numbers λ belong to the resolvent set of $A(t)$, the same is true for $\bar{A}(t)$, and, in particular, $[I-\bar{\bar{A}}(t)]^{-1}$ exists. $B(t)$ and $B(t)^{-1}$ are continuous in t in the uniform topology, so that there exists a constant M such that $||B(t)|| \leq M$, $||B(t)^{-1}|| \leq M$. Moreover, $B(t)$ is of bounded variation in the sense that

$$(4.1) \qquad \sum_{j=1}^{n} ||B(t_j) - B(t_{j-1})|| \leq \int_a^b ||\dot{B}(t)||dt \equiv N$$

for any partition $a = t_o < t_1 < \dots < t_n = b$ of the interval $a \leq t \leq b$. Again, it follows that

$$\left[I - \bar{\bar{A}}(t)\right]^{-1} = \left[I - \bar{\bar{A}}(a)\right]^{-1} B(t)^{-1}$$

and

$$\left[I - A(t)\right]^{-1} = R(t)^{-1}\left[I - \bar{\bar{A}}(t)\right]^{-1} R(t)$$

are continuous in t in the uniform topology.

We can now state our main theorem.

Theorem 2: Under Assumptions 1 and 2, there exists a unique operator-valued function $U(t,s)$, defined in the triangle $a \leq s \leq t \leq b$, with the following properties:

(4.2) $\left\{\begin{array}{l}U(t,s) \text{ is a bounded linear operator and is strongly continuous} \\ \text{in } (s,t), \text{ with } U(t,t) = I;\end{array}\right.$

(4.3) $U(t,r) = U(t,s)U(s,r), \qquad r \leq s \leq t;$

(4.4) $U(t,s)$ maps $\mathcal{D}[A(s)]$ into $\mathcal{D}[A(t)]$, and for each $u_o \in \mathcal{D}[A(s)]$,

$$u(t) = U(t,s)u_o$$

is strongly differentiable for $t \geq s$ and satisfies (1.1a); moreover $A(t)u(t)$ is strongly continuous in t.

In particular $u(t) = U(t,s)u(a)$ satisfies the homogeneous equation (1.1a) for each $u(a) \in \mathcal{D}[A(a)]$.

The solution of the inhomogeneous equation (1.1) can be expressed in terms of the operator U(t,s) formally as follows:

$$u(t) = U(t,a)u(a) + \int_a^t U(t,s)f(s)ds. \tag{4.5}$$

As to the validity of this expression as the solution of (1.1), we have

Theorem 3: (4.5) is the solution of (1.1) if $u(a) \in \mathcal{D}[A(a)]$ and if the function f(s) satisfies one of the following conditions:

(4.6) f(t) is strongly continuously differentiable;

or

(4.7) $f(t) \in \mathcal{D}[A(t)]$, and A(t)f(t) and f(t) are strongly continuous in t.

The proofs of these theorems, which are rather complicated, will be sketched here; they are like those given in the earlier paper [3]. The problem can, without loss of generality, be reduced to the case R(t) = I. It is then similar to the problem treated in the earlier paper [3], except that now we have $A(t) \in (S')$ instead of $A(t) \in (S)$. The operator U(t,s) is constructed by a Riemann product integration method, and the properties (4.2) and (4.3) are proved. The proof of (4.4) is the most complicated step; it is effected by showing that the operator $W(t,s) = [I-A(t)]U(t,s)[I-A(s)]^{-1}$ is bounded. To do this, a limiting procedure based on the approximation of U(t,s) by a finite 'Riemann product' must be used. To prove Theorem 3 we must first show that

$$\int_a^t U(t,s)f(s)ds$$

belongs to $\mathcal{D}[A(t)]$. This requires some calculation in the case where condition (4.6) is assumed to hold.

5. Examples

Rather than discuss the applications of our results in detail, we shall illustrate them by some simple but non-trivial examples.

First, consider the Schroedinger equation

$$(5.1)\qquad -\frac{1}{i}\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left[p(x,t)\frac{\partial u}{\partial x}\right] + q(x,t)u, \quad 0 \le x \le 1, \quad a \le t \le b.$$

Here the unknown $u = u(x,t)$ is a function of two variables x and t. We assume that the coefficients p, q are smooth real-valued functions of x, t and that $p > 0$; further, the time-dependent homogeneous boundary conditions

$$(5.2)\qquad u_x(0,t) - \alpha(t)u(0,t) = 0, \qquad u_x(1,t) + \beta(t)u(1,t) = 0$$

are imposed at the endpoints of the interval $[0,1]$. In (5.2), $\alpha(t)$, $\beta(t)$ are smooth real-valued functions.

Eq. (5.1) may be written in the form (1.1a) where $A(t)$ is, for each t, a second-order differential operator with the boundary conditions (5.2). If we take as X the Hilbert space $L^2(0,1)$, then $A(t)$ can be written in the form $A(t) = -iH(t)$ with the operator $H(t)$ self-adjoint. Thus Assumption 1 is satisfied simply with $Q(t) = I$.

It can be shown that Assumption 2 is also satisfied. The domain of $H(t)$ consists of all functions of $L^2(0,1)$ whose second derivatives also belong to $L^2(0,1)$ and which satisfy the boundary conditions (5.2). Thus it is easy to find the operator $R(t)$ required in Assumption 2. We can take as $R(t)$ a multiplicative operator

$$(5.3)\qquad R(t) = \exp[r(x,t)],$$

where $r(s,t)$ is any smooth real-valued function such that

$$r_x(0,t) + \alpha(t) = 0, \qquad r_x(1,t) - \beta(t) = 0. \tag{5.4}$$

For instance, we may take

$$r(x,t) = -\alpha(t)x + \frac{1}{2}\left[\alpha(t) + \beta(t)\right] x^2 .$$

Then R(t) maps the domain of H(t) onto the set $\mathcal{D}$ of all functions $u(x) \in L^2(0,1)$, with $u_{xx} \in L^2(0,1)$, which satisfy the boundary conditions $u_x(0) = u_x(1) = 0$. (Thus $\mathcal{D}$ is independent of t.) It is easy to see that the remaining conditions of Assumption 2 are satisfied.

Thus (5.1), interpreted as a differential equation in $X = L^2(0,1)$, has one and only one solution with a given initial value $u(a) \in \mathcal{D}\,[H(a)]$.

It is true that Assumption 2 is still rather restrictive, and much would be gained if it could be replaced by a weaker assumption. A natural question is whether it is sufficient to require that the operator $[I-A(t)]^{-1}$, which is bounded by Assumption 1, be, say, strongly continuously differentiable. But the following example shows that this is not the case. Consider the Schroedinger equation

$$\frac{du(t)}{dt} = -i\, H(t)u(t) \tag{5.5}$$

in the Hilbert space $X = L^2(-\infty,\infty)$, where H(t) is a multiplicative operator

$$H(t) = -(x-t)^{-2}. \tag{5.6}$$

Then the solution of (5.5), if it exists, could not be other than

$$u(t) = u(x,t) = \exp\left(\frac{i}{x-t} - \frac{i}{x-a} \right) u(x,a).$$

But this does not satisfy (1.1) as an operational equation in our sense, unless $u(x,a)$ were to vanish identically in the interval $a \le x \le b$, because otherwise $u(t)$ does not belong to the domain of H(t) for all t, $a \le t \le b$. Such $u(x,a)$ are very special functions and do not form a dense set in X. Nevertheless, the operator $H(t)$ is self-adjoint and changes rather smoothly in the sense that $[I-iH(t)]^{-1}$ is not only continuously differentiable but also analytic in t.

It is interesting to note that, even in this example, we can find a bounded operator function $R(t)$ with the property that $\bar{\bar{H}}(t) = R(t)H(t)R(t)^{-1}$ has a constant domain. We need only take as $R(t)$ the unitary operator which maps $u(x)$ into $u(x+t)$. But then $\dot{R}(t)$ is not a bounded operator, so that the Assumption 2 is not satisfied.

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